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# Eliminating the Coriolis Effect in Liquid Mirrors

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## ABSTRACT

If uncorrected, the Coriolis force due to the rotation of the Earth causes significant aberration of images produced by large liquid-mirror telescopes. We show that this problem can be eliminated by a fixed compensating tilt of the liquid-mirror rotation axis. The required tilt angle, which is a function of latitude and mirror rotation rate, is of order 10 arcsec for current telescopes. This result removes the last fundamental obstacle to achieving diffraction-limited performance with large liquid mirrors.

*Subject headings:* telescopes

## 1. Introduction

Liquid-mirror telescopes employ a rotating primary mirror surfaced with a metallic liquid, usually mercury, to reflect and focus light (Borra 1982; Hickson et al. 1994; Potter and Mulrooney 1997). Laboratory tests and astronomical observations (Borra et al. 1992; Hickson and Mulrooney 1998, 2001) indicate that liquid mirrors can provide an optical surface that is parabolic to within a fraction of a wave and provide astronomical-quality images. Current liquid-mirror telescopes are zenith-pointing, however, their comparatively low cost makes them competitive with conventional telescopes for many types of observations such as wide-angle surveys.

Departures from the ideal parabolic shape, due to various influences such as gravitational nonuniformity and Lunar tides, have been analysed by Gibson and Hickson (1992) and Mulrooney (2000). These authors found that these influences have a negligible effect on the images produced by liquid mirror telescopes with the exception of the Coriolis force that results from the Earth's rotation. They found that, for an ideal non-viscous fluid, the

Coriolis effect introduces astigmatism and coma that results in a image spread that could be as large as  $\sim 1.7$  arcsec for a 2.7m f/1.5 telescope to  $\sim 6.7$  arcsec for a 10m f/1.5 telescope. This is clearly a very significant effect, potentially being larger even than the atmospheric seeing.

While the Coriolis aberrations might in principle be removable by a suitably-designed optical corrector, the exact correction required is difficult to determine because of hydrodynamic and viscous effects in the fluid layer that covers the mirror. The presence of any non-axisymmetric force will cause a periodic time-dependent flow of the fluid. In addition to greatly complicating the modeling of the system, such a flow will result in print-through of any imperfections in the surface of the structure that supports the fluid layer, making complete correction of the images very difficult.

In this paper it is shown that the Coriolis effect can in fact be eliminated by means of a tilt of the rotation axis of the mirror. When this is done, there are no significant time-dependent forces in the rotating frame of the mirror, and the fluid surface is static. Because there is no fluid flow there are no hydrodynamic effects and the mirror's surface is a paraboloid to within a small fraction of the wavelength of light.

## 2. Eliminating the Coriolis effect

In the analysis that follows, we assume that sufficient time has elapsed from the start of rotation to allow the fluid to come to equilibrium. It is then sufficient to consider only inertial and gravitational forces; other effects such as surface tension have been shown by direct laboratory tests (Borra et al. (1992)) to have a negligible effect on the shape of the surface. We will justify the assumption of equilibrium by showing that, when the mirror rotation axis is chosen appropriately, there are no significant time-dependent forces acting on the fluid that would cause a departure from equilibrium.

Consider a liquid mirror telescope located at latitude  $l$ , The axis and rate of rotation of the liquid mirror is defined by the angular velocity vector  $\boldsymbol{\Omega}_0$ . In the rotating frame of the Earth, the instantaneous velocity of an element of fluid is given by

$$\mathbf{v}_0 = \boldsymbol{\Omega}_0 \times \mathbf{r}, \tag{1}$$

where  $\mathbf{r}$  is the vector extending from the vertex of the mirror (the point where the rotation axis intersects the surface of the fluid) to the fluid element.

Now consider an inertial frame moving, but not rotating, with the Earth. We can ignore the small acceleration due to the Earth's motion around the Sun. The resulting centrifugal

force is balanced by the Sun’s gravitational attraction, and it has already been shown that the tidal force of the Moon, and therefore the smaller tidal force of the Sun, are unimportant (Gibson and Hickson 1992). In the inertial frame, the fluid is also rotating about the Earth’s axis with velocity

$$\mathbf{v} = \boldsymbol{\Omega}_{\oplus} \times (\mathbf{R}_{\oplus} + \mathbf{r}), \quad (2)$$

where  $\boldsymbol{\Omega}_{\oplus}$  is the angular velocity vector of the rotating Earth and  $\mathbf{R}_{\oplus}$  is the vector extending from the center of the Earth to the vertex of the mirror. The velocity of the fluid element in the inertial frame is thus

$$\begin{aligned} \mathbf{v}_e &= \mathbf{v}_0 + \mathbf{v} \\ &= \boldsymbol{\Omega}_e \times \mathbf{r} + \boldsymbol{\Omega}_{\oplus} \times \mathbf{R}_{\oplus}, \end{aligned} \quad (3)$$

where

$$\boldsymbol{\Omega}_e = \boldsymbol{\Omega}_0 + \boldsymbol{\Omega}_{\oplus} \quad (4)$$

is the effective angular velocity vector.

Eqn (3) shows that the effect of the Earth’s rotation is twofold. First, the effective axis and rate of rotation is changed to  $\boldsymbol{\Omega}$ , the vector sum of  $\boldsymbol{\Omega}_0$  and  $\boldsymbol{\Omega}_{\oplus}$ . Second, an acceleration

$$\begin{aligned} \mathbf{a}_C &= \frac{d\mathbf{V}}{dt} \\ &= \boldsymbol{\Omega}_{\oplus} \times (\boldsymbol{\Omega}_{\oplus} \times \mathbf{R}_{\oplus}) \end{aligned} \quad (5)$$

occurs, directed inward toward the Earth’s axis, with magnitude

$$a_C = R_{\oplus} \Omega_{\oplus}^2 \cos l. \quad (6)$$

Consider now the gravitational force acting on the fluid. The gravitational acceleration is

$$\begin{aligned} \mathbf{g} &= -\frac{GM_{\oplus}}{|\mathbf{R}_{\oplus} + \mathbf{r}|^3} (\mathbf{R}_{\oplus} + \mathbf{r}) \\ &= \mathbf{g}_0 + \mathbf{g}_T, \end{aligned} \quad (7)$$

where

$$\mathbf{g}_0 = -\frac{GM_{\oplus}}{R_{\oplus}^3} \mathbf{R}_{\oplus} \quad (8)$$

is the gravitational acceleration at the mirror’s vertex and

$$\mathbf{g}_T = -\frac{GM_{\oplus}}{|\mathbf{R}_{\oplus} + \mathbf{r}|^3} (\mathbf{R}_{\oplus} + \mathbf{r}) - \mathbf{g}_0 \quad (9)$$

is a small tidal term representing the differential acceleration across the mirror due to the gradient and divergence of the gravitational field.

The total centripetal acceleration of the fluid element must equal the gravitational acceleration, which is the only significant external force on the fluid. Therefore,

$$\boldsymbol{\Omega}_e \times (\boldsymbol{\Omega}_e \times \mathbf{r}) + \mathbf{a}_C = \mathbf{g}_0 + \mathbf{g}_T \quad (10)$$

Ignoring for the moment the small tidal term  $\mathbf{g}_T$ , to which we shall return later, it is now evident that if the total angular velocity vector  $\boldsymbol{\Omega}$  is parallel to the effective gravitational acceleration  $\mathbf{g}_e = \mathbf{g}_0 - \mathbf{a}_C$ , the force acting on the fluid will be symmetric about the axis defined by  $\boldsymbol{\Omega}$ . To see this observe that, when the fluid is in equilibrium, the radius  $r$  of any fluid element is constant, so the centripetal acceleration  $\boldsymbol{\Omega}_e \times (\boldsymbol{\Omega}_e \times \mathbf{r})$  has constant magnitude and is directed toward the rotation axis. By definition, the effective gravitational acceleration  $\mathbf{g}_e$  acts along the rotation axis, so the fluid element feels only constant radial and axial forces, and no azimuthal forces. Because these forces are constant, there will be no further flow of the fluid once it has reached equilibrium and therefore no hydrodynamic effects. The required geometrical condition is illustrated in Fig. 1. Since we are free to choose the orientation of the mirror rotation axis, it is always possible to achieve this condition. *This shows that the Coriolis effect can be eliminated by tilting the mirror rotation axis in order to make the effective angular velocity vector  $\boldsymbol{\Omega}_0 + \boldsymbol{\Omega}_\oplus$  parallel to the effective acceleration vector  $\mathbf{g}_0 - \mathbf{a}_C$ .*

Referring to Fig. 1, we see that the required tilt angle, with respect to the zenth, is  $\alpha - \beta$  where

$$\begin{aligned} \alpha &= \frac{\mathbf{g}_e \cdot \mathbf{g}_0}{g_e g_0} \\ &\approx \frac{\Omega_\oplus^2 R_\oplus \sin l}{g_0} \end{aligned} \quad (11)$$

and

$$\begin{aligned} \beta &= \frac{\boldsymbol{\Omega}_e \cdot \boldsymbol{\Omega}_0}{\Omega_e \Omega_0} \\ &\approx \frac{\Omega_\oplus \cos l}{\Omega_0} \end{aligned} \quad (12)$$

In practice, liquid mirrors are leveled using a precise bubble level which is of course sensitive to the effective vertical direction defined by  $\mathbf{g}_e$ , not the true vertical. In this case, the required tilt angle is just  $\beta$ . From Fig. 1. it is evident that the required axis tilt is in the North-South direction and towards the equator if the mirror rotates in the same sense

as the Earth, (counterclockwise in the northern hemisphere and clockwise in the southern hemisphere) and away from the equator otherwise.

Now,  $\Omega_{\oplus} \approx 7.29 \times 10^{-5} \text{ s}^{-1}$ ,  $R_{\oplus} \approx 6.38 \times 10^6 \text{ m}$ ,  $g_0 \approx 9.81 \text{ ms}^{-2}$  and  $\Omega_0^2 \approx (g_0/2F)$ , where  $F$  is the telescope focal length. Substituting these quantities in Eqns (11) and (12) gives

$$\alpha \approx 3.46 \times 10^{-3} \sin l \quad (13)$$

$$\beta \approx 3.29 \times 10^{-5} F^{1/2} \cos l \quad (14)$$

where  $F$  is measured in meters.

### 3. The equilibrium surface and the effect of the tidal field

Let us now assume that the Coriolis-cancelling condition described in the previous section is satisfied. What is the shape of the mirror's surface? To calculate this, it is convenient to use cylindrical coordinates  $(\rho, \phi, z)$  with the  $z$  axis aligned with the effective rotation axis  $\mathbf{\Omega}_e$ , and origin at the vertex of the mirror. The orientation is chosen so that  $z$  increases in the outward direction (ie. away from the Earth) and  $\phi = 0$  is the North direction.

Rather than work with the vector equation (10), we introduce the scalar potential  $\Phi$ , whose gradient is the acceleration. The total potential of a fluid element is then

$$\Phi = -\frac{1}{2}\Omega_e^2 \rho^2 + g_e z + \Phi_T. \quad (15)$$

In this equation, the first term on the right hand side is the centrifugal potential arising from rotation about the axis  $\mathbf{\Omega}_e$ , the second term represents the effective gravitational acceleration and the third term is the potential corresponding to the tidal term  $\mathbf{g}_T$ . The zero point of the potential is arbitrary because any constant can be added without affecting the acceleration. We have chosen  $\Phi = 0$  at the mirror vertex ( $\rho = mz = 0$ ).

In equilibrium, the surface of a liquid has constant potential, so Eqn (15) gives the surface shape directly,

$$z = \frac{\Omega_e^2}{2g_e} \rho^2 - \frac{1}{g_e} \Phi_T \quad (16)$$

which is a paraboloid of focal length

$$F = \frac{1}{4} \left( \frac{dz}{d\rho^2} \right)^{-1} = \frac{g_e}{2\Omega_e^2} \quad (17)$$

with a small correction due to the tidal term. The quantities  $\Omega_e$  and  $g_e$ , the effective angular velocity and gravitational acceleration, can be determined by applying the cosine rule to the geometry of Fig 1. and using Eqn (6). The result is

$$\Omega_e = [\Omega_0^2 + \Omega_\oplus^2 + 2\Omega_0\Omega_\oplus \sin(l - \epsilon)]^{1/2} \quad (18)$$

$$g_e = [g_0^2 + (\Omega_\oplus^4 R_\oplus^2 - 2g_0\Omega_\oplus^2 R_\oplus) \cos^2 l]^{1/2} \quad (19)$$

Let us now determine the effect of the tidal term. Eqn (9) shows that the tidal term represents the difference between the actual gravitational acceleration and a hypothetical uniform gravitational acceleration equal to the value at the mirror vertex. However, because of the Earth's rotation, the effective axis of rotation of the mirror does not pass through the centre of the Earth, but is instead tilted from the vertical. Thus, there is a tilt between our cylindrical coordinate system and a geocentric system. To first order in the tilt angle  $\alpha$ , the tidal potential is therefore

$$\Phi_T = -\frac{GM_\oplus}{|\mathbf{R}_\oplus + \mathbf{r}|} - \frac{GM_\oplus}{R_\oplus^2}(z - \alpha\rho \cos \phi) + \frac{GM_\oplus}{R_\oplus} \quad (20)$$

where the last term is a constant chosen to make  $\Phi_T = 0$  at the mirror vertex. To first order in  $\alpha$ , we have

$$|\mathbf{R}_\oplus + \mathbf{r}| = [(R_\oplus + z - \alpha\rho \cos \phi)^2 + \rho^2]^{1/2} \quad (21)$$

Using Eqn (21), the first term on the right-hand side of Eqn (20) can be expanded in a power series in  $\rho/R_\oplus$  and  $z/R_\oplus$  and  $z$  can be eliminated by substituting Eqn (16). From Eqn (16), dividing this result by  $-g_e$  then gives the resulting surface deformation  $z_T$ . To fourth order in  $\rho$  and first order in  $\alpha$ , the result is

$$z_T = \frac{g_0}{g_e} \left\{ \frac{\rho^2}{2R_\oplus} - \left[ 6 + 6\frac{R_\oplus}{F} + \frac{R_\oplus^2}{F^2} \right] \frac{\rho^4}{16R_\oplus^3} - \left[ 3 + \frac{R_\oplus}{F} \right] \frac{\alpha\rho^3 \cos \phi}{2R_\oplus^2} \right\} \quad (22)$$

The first two terms in this equation are already known and correspond to a small shift in focal length and a small amount of spherical aberration. The new focal length (from Eqns 17, 16 and 22) is

$$F_e = F \left[ 1 + \frac{2F}{R_\oplus} \right]^{-1}. \quad (23)$$

The maximum amplitude of the second term is approximately  $D^4/256F^2R_\oplus$ , where  $D$  is the diameter of the mirror. For a 10m f/2.5 mirror it is  $10^{-7}$  m. At a wavelength of 1  $\mu$ m, this corresponds to a fifth of a wave of spherical aberration, which can easily be corrected optically if necessary.

The third term is a very small comatic distortion that arises from the misalignment of the tidal field with the effective rotation axis. The maximum amplitude of this term

is  $\alpha D^3/16FR_\oplus$ . For a 10-m f/2.5 mirror located at 30 degrees latitude it has the value  $6.8 \times 10^{-10}$  m, which is less than a thousandth of a wave. Even for a 100-m telescope the maximum error is less than a tenth of a wave. It follows that this term can be safely ignored for all liquid-mirror telescopes currently conceivable.

#### 4. Discussion

Until now, the Coriolis effect was considered to be a potentially significant problem for large liquid-mirror telescopes. While viscous effects might reduce the the surface distortion to some degree, the predicted effect is so large for 10-m class mirrors that it was a cause of concern.

In this paper we have shown that the Coriolis effect can in fact be eliminated by a fixed compensating tilt of the axis of rotation of the liquid mirror. A small residual tidal effect was shown to be of no significance to all present and forseen liquid-mirror telescopes. This result removes the last fundamental obstacle to achieving diffraction-limited performance with large liquid mirrors.

In order to eliminate the Coriolis effect, the axis of rotation of the mirror must be tilted from the effective vertical (defined by a pendulum or bubble level) by the angle  $\beta$  given by Eqns (12) and (14). For the 3-m f/1.5 mirror of the Nasa Orbital Debris Observatory (Potter and Mulrooney 1997), which rotates counterclockwise, the required shift is to the South by an angle of 12.1 arcsec. Axis tilts of this magnitude have been made with this telescope, in order to investigate the effects on the image (Mulrooney 2000), but the effects were smaller than the atmospheric seeing and could not be adequately assessed. For the 6-m f/1.5 mirror of the Large Zenith Telescope (Hickson et al. 1998), the tilt required is 13.1 arcsec. The larger mirror diameter and better image sampling should allow us to verify the technique using this telescope.

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## REFERENCES

- Borra, E. F. 1982, JRASC, 76, 246
- Borra, E. F., Content, R., Girard, L., Szapiel, S., Tremblay, L. M., Boily, E 1992, ApJ, 392, 375
- Gibson, B. K. and Hickson, P. 1992, ApJ, 391, 409
- Hickson, P., Borra, E. F., Cabanac, R., Chapman, S. C., del Lapparent, V., Mulrooney, M. and Walker, G. A. H. 1998, Pub. Soc. Photoptical Instr. Eng., 3352, 226
- Hickson, P., Borra, E. F., Cabanac, R., Content, R., Gibson, B. K., and Walker, G. A. H. 1994, ApJ, 436, L201
- Hickson, P. and Mulrooney, M. K. 1998, ApJS, 115, 35
- Hickson, P. and Mulrooney, M. K. 2001, in preparation
- Potter, A. and Mulrooney, M. K. 1997, Adv. Sp. Sc., 19, 213
- Mulrooney, M. K. 2000, Ph.D. Thesis, Rice Univ.



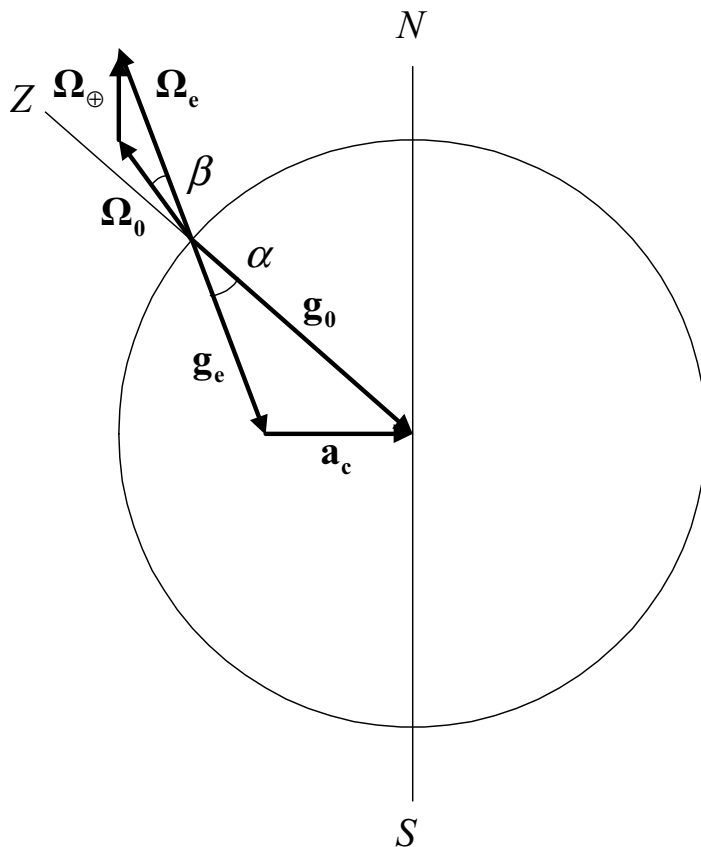


Fig. 1.— Geometry of angular velocity and acceleration vectors. The circle represents the Earth, rotating about the N-S axis with angular velocity  $\Omega_{\oplus}$ . The vertex of the liquid mirror is located at the intersection of the circle and the straight line extending to the zenith Z. In the rotating frame of the Earth, the mirror rotates about axis  $\Omega_0$  but in an inertial frame it rotates about the resultant axis  $\Omega_e$ . To eliminate the Coriolis effect, this axis must be parallel to the effective gravitational acceleration  $\mathbf{g}_e$